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STABILITY ANALYSIS OF ANNULAR FLOW STRUCTURE, USING A DISCRETE FORM OF THE "TWO-FLUID MODEL"

D. Barnea
Department of Fluid Mechanics and Heat Transfer, Faculty of Engineering, Tel-Aviv University, Ramat-Aviv 69978, Israel

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Abstract—The differential form of the "two-fluid model" for annular flow, neglecting surface tension, is ill-posed, and it is not suited for examining the stability of the steady-state solutions with respect to the average film thickness. It is shown here that a discrete (difference) representation of the two-fluid model may lead to an appropriate criterion for the stability of the steady-state solutions. Exactly the same criterion is obtained from the requirement that the kinematic waves will propagate in the downstream direction. The suggested discrete form of the "two-fluid model" is used to perform transient simulation and for examining the system response to finite disturbances.

Key Words: two phase flow, annular flow, stability

INTRODUCTION

The simplest and most effective method for calculating pressure drop and film thickness for steady-state annular flow is to use the integral approach which treats the gas and liquid as two separated fluids with a spatially uniform cross-sectional area. Effective wall and interfacial shear stresses are required to account for the wavy structure of the liquid film. The solution of this steady-state model may yield multiple steady-state solutions, some of which are unstable.

The transient formulation which is consistent with the aforementioned steady-state solutions is provided by the "two-fluid model" in which surface tension and gravity are neglected. This transient equation system of the two-fluid model possesses complex-valued characteristics and thus, as is well known, it is not well-posed as an initial value problem. The complex characteristics associated with these equations imply unbounded exponential growth (Lyczkowski *et al.* 1978; Ramshaw & Trapp 1978; Stewart & Wendroff 1984; Jones & Prosperetti 1987; Lin & Hanratty 1986; Banerjee 1985). This ill-posedness manifests itself as instability of the differential equations and indeed, a linear stability analysis of annular flow using the two-fluid model, neglecting surface tension, shows, as is well known, that the steady-state solutions are always unstable owing to the Kelvin–Helmholtz (K–H) type instability (Ramshaw & Trapp 1978; Barnea & Taitel 1989).

Barnea & Taitel (1989, 1990) claimed that the full two-fluid model is not suitable to examine the stability of steady annular flow, where the solution is obtained for an average film thickness using effective interfacial shear stresses. This is because the existence of K–H instability in steady annular flow is expected *a priori* and steady annular flow is dynamically unstable with regard to its interface, resulting in an unstable wavy interface. However, our question of interest is whether this kind of steady annular flow is a stable structure with respect to its average film thickness and average phase velocities as obtained in the steady-state solutions. In order to answer this question Barnea & Taitel (1989) suggested a few simplified dynamic formulations based on various assumptions. Common to all assumptions is that all of them eliminate the Bernoulli amplification from the transient equations. Using these dynamic formulations a linear stability analysis was performed, arriving at a simple logical criterion for the stability of the multiple steady-state solutions. It seems that these dynamic formulations are those that are suited to answer the question whether annular flow with an unstable wavy interface is a stable structure that can be realized physically.

In the present work it will be shown that some results obtained from the viscous K–H analysis, on the full two-fluid model, can be also used to arrive at the same criterion for stable steady-state solutions, as that obtained by the simplified transient formulations. In addition it will be shown that the two-fluid model presented in a spatial discrete form ("the cell model") also leads to the aforementioned structure stability. Thus, although the transient two-fluid model with the elimination of gravity and surface tension is ill-posed, it can be used to obtain logical transient simulations which converge to the steady state which is stable according to the criterion presented by Barnea & Taitel (1989).

LINEAR STABILITY ANALYSIS USING THE TWO-FLUID MODEL

The transient two-fluid model for annular flow, assuming incompressible flow and neglecting surface tension is formulated by the following equations:

$$\frac{\partial \varepsilon_L}{\partial t} + \frac{\partial (\varepsilon_L U_L)}{\partial x} = 0, \qquad [1]$$

$$\frac{\partial \varepsilon_G}{\partial t} + \frac{\partial (\varepsilon_G U_G)}{\partial x} = 0, \qquad [2]$$

$$\rho_L \frac{\partial}{\partial t}(\varepsilon_L U_L) + \rho_L \frac{\partial}{\partial x}(\varepsilon_L U_L^2) + \varepsilon_L \frac{\partial P}{\partial x} = -\frac{\tau_L S_L}{A} + \frac{\tau_i S_i}{A} - \rho_L \varepsilon_L g \sin\beta \qquad [3]$$

and

$$\rho_G \frac{\partial}{\partial t}(\varepsilon_G U_G) + \rho_G \frac{\partial}{\partial x}(\varepsilon_G U_G^2) + \varepsilon_G \frac{\partial P}{\partial x} = -\frac{\tau_i S_i}{A} - \rho_G \varepsilon_G g \sin\beta, \qquad [4]$$

where P is the pressure, t is time, x is the coordinate in the flow direction, g is the acceleration of gravity, A is the cross-sectional area, U is the axial average velocity, τ is the wall shear stress, τ_i is the interfacial shear stress, S and S_i are the perimeters over which τ and τ_i act, ε is the phase holdup, ρ is the phase density and β is the angle of inclination from the horizontal (positive for upward flow). The subscripts L and G denote liquid and gas, respectively.

Eliminating the pressure drop from [3] and [4] yields

$$\rho_L \frac{\partial U_L}{\partial t} - \rho_G \frac{\partial U_G}{\partial t} + \rho_L U_L \frac{\partial U_L}{\partial x} - \rho_G U_G \frac{\partial U_G}{\partial x} = F, \qquad [5]$$

where

$$F = \frac{S_i}{A}\left(\frac{1}{\varepsilon_L} + \frac{1}{\varepsilon_G}\right)(\tau_i - \tau_{iL}), \qquad [6]$$

τ_{iL} is the interfacial shear stress needed for a steady-state flow for a given liquid velocity, U_L, and film thickness, δ,

$$\tau_{iL} = \frac{(\rho_L - \rho_G) A g \sin\beta + \dfrac{\tau_L S_L}{\varepsilon_L}}{S_i\left(\dfrac{1}{\varepsilon_G} + \dfrac{1}{\varepsilon_L}\right)}, \qquad [7]$$

and τ_i is the shear stress provided by the gas, which depends on the gas velocity, U_G, and the film thickness, δ,

$$\tau_i = \tfrac{1}{2} f_i \rho_G U_G^2. \qquad [8]$$

For steady state $\tau_{iL} = \tau_i$.

A linear stability analysis of [1], [2] and [5] yields the following criterion for stability (Barnea & Taitel 1989):

$$(C_R - a)^2 + \frac{\rho_L \rho_G}{\rho^2 \varepsilon_L \varepsilon_G}(U_G - U_L)^2 < 0, \qquad [9]$$

where

$$a = \frac{\rho_L U_L \varepsilon_G + \rho_G U_G \varepsilon_L}{\rho_L \varepsilon_G + \rho_G \varepsilon_L}, \qquad \rho = \frac{\rho_L}{\varepsilon_L} + \frac{\rho_G}{\varepsilon_G} \tag{10a,b}$$

and

$$C_R = \frac{\left(\frac{\partial F}{\partial \varepsilon_L}\right)_{U_{LS}, U_{GS}}}{\left(\frac{\partial F}{\partial U_{GS}}\right)_{U_{LS}, \varepsilon_L} - \left(\frac{\partial F}{\partial U_{LS}}\right)_{U_{GS}, \varepsilon_L}}. \tag{11}$$

The term C_R corresponds to the wave velocity at marginal stability (Barnea & Taitel 1989; Barnea 1991).

It is interesting to observe that the expression for C_R is also exactly the same as the kinematic wave velocity presented by Wallis (1969) and Wu *et al.* (1987):

$$C_K = \left.\frac{\partial U_{LS}}{\partial \varepsilon_L}\right|_{U_{LS} + U_{GS}} = C_R. \tag{12}$$

Criterion [9] indicates that the flow is always unstable, since both terms on the l.h.s. are always positive. This criterion is related to the well-known K–H instability when the shear stresses are taken into account. It has been shown (Andreussi *et al.* 1985; Barnea 1991) that as long as the liquid supply in the film is sufficiently small to prevent blockage of the gas core the instability represented in [9] results in large-amplitude roll waves on the film interface. Namely, criterion [9] indicates that the interface of steady annular flow is always unstable.

However, our question of interest is whether this steady annular flow with its unstable wavy interface is a stable structure with respect to the average film thickness obtained in the steady-state solution. It is assumed here that in order to maintain the stable structure of annular flow the characteristic velocity, C_K, must be compatible with upstream boundary conditions, namely for co-current flow $C_K(=C_R)$ should be positive. In the case when the kinematic waves are in the opposite direction the general structure of the wavy liquid film will collapse, namely the steady-state solution that was obtained for the average film thickness is physically unstable and will not exist. Thus, for co-current gas–liquid flow where the liquid film velocity is positive the condition for structural stability is:

$$C_K = C_R > 0. \tag{13}$$

Since the denominator in [11] is always positive, the stability criterion for co-current flow is

$$\left.\frac{\partial F}{\partial \varepsilon_L}\right|_{U_{LS}, U_{GS}} > 0 \tag{14}$$

or

$$\left.\frac{\partial \tau_{iL}}{\partial \varepsilon_L}\right|_{U_{LS}} - \left.\frac{\partial \tau_i}{\partial \varepsilon_L}\right|_{U_{GS}} < 0. \tag{15}$$

For the case of counter-current flow where U_{LS} is negative a similar derivation yields:

$$\left.\frac{\partial \tau_{iL}}{\partial \varepsilon_L}\right|_{U_{LS}} - \left.\frac{\partial \tau_i}{\partial \varepsilon_L}\right|_{U_{GS}} > 0. \tag{16}$$

These are exactly the same criteria as those obtained by Barnea & Taitel (1989) who used quite different considerations and performed a linear stability analysis on a simplified transient formulation.

Equations [15, 16] provide simple and general criteria for determining the linear stability of the steady-state solutions. Note again, that these criteria point out whether the steady-state solution is a stable structure with respect to its average film thickness. These criteria are easily applied by using steady-state diagrams where τ_i is plotted vs δ/D; D is the pipe diameter. Figure 1 is an

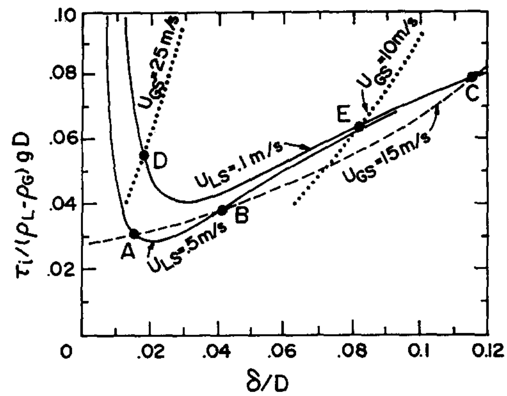

Figure 1. Steady-state solutions and stability diagram. Air–liquid, 0.1 MPa, 25°C, 2.5 cm dia. ——, τ_{iL} [7]; - - -, τ_i [8], $f_i = 0.05$; . . . , τ_i [8], $f_i = 0.005\ (1 + 300\,\delta/D)$.

example of such a diagram for vertical co-current annular flow, where two different correlations for the interfacial shear stresses, τ_i, are used. When a constant value of the interfacial friction factor f_i is used, multiple solutions may occur: two linearly stable solutions (points A and C) and one unstable solution (point B). When Wallis's (1969) correlation is used for f_i, a single linearly stable solution is obtained for any gas and liquid flow rate (points D or E). Barnea & Taitel (1990) have shown that only solutions on the solid lines that are to the left of the minimum of the τ_{iL} vs δ curve are stable to finite disturbances.

In the next section it will be shown that a linear stability analysis of the discrete representation of the "two-fluid model" will lead to exactly the same stability criterion for the annular structure as that in [15] and [16].

THE DISCRETE REPRESENTATION—THE CELL MODEL

The discrete form of the set of partial differential equations [1]–[4], for the special case of the quasi-equilibrium assumption in the gas, is established by dividing the pipe into n equal-sized cells. Continuity and momentum balances are applied to each cell resulting in $2n$ ordinary differential equations. The equations are derived with the advective terms approximated by "backward" difference operators, namely information is travelling only in the downstream direction.

The resulting difference equations are (see the appendix):

$$\frac{d\varepsilon_{L1}}{dt} = \frac{1}{l}\,(U_{LS} - \varepsilon_{L1}\,U_{L1}) \qquad [17]$$

and

$$\frac{dU_{L1}}{dt} = \frac{1}{l\varepsilon_{L1}}\left(\frac{U_{LS}^2}{\varepsilon_{L1}} - U_{LS}\,U_{L1}\right) + \frac{4}{\rho_L D\varepsilon_{L1}\sqrt{1-\varepsilon_{L1}}}\,(\tau_{i1} - \tau_{iL1}); \qquad [18]$$

and for $j = 2, \ldots, n$,

$$\frac{d\varepsilon_{Lj}}{dt} = \frac{1}{l}\,(U_{Lj-1}\,\varepsilon_{Lj-1} - U_{Lj}\,\varepsilon_{Lj}) \qquad [19]$$

and

$$\frac{dU_{Lj}}{dt} = \frac{\varepsilon_{Lj-1}}{l\varepsilon_{Lj}}\,(U_{Lj-1}^2 - U_{Lj-1}\,U_{Lj}) + \frac{\rho_G}{\rho_L}\frac{U_{GS}^2}{l}\left[\frac{1}{(1-\varepsilon_{Lj})^2} - \frac{1}{(1-\varepsilon_{Lj-1})(1-\varepsilon_{Lj})}\right] + \frac{4}{\rho_L D\varepsilon_{Lj}\sqrt{1-\varepsilon_{Lj}}}\,(\tau_{ij} - \tau_{iLj}). \qquad [20]$$

Equations [17]–[20] are in the general form

$$\left.\begin{aligned} &\frac{d\varepsilon_{L1}}{dt}=f_1(\varepsilon_{L1},U_{L1}), && \frac{dU_{L1}}{dt}=g_1(\varepsilon_{L1},U_{L1}),\\ &\frac{d\varepsilon_{Lj}}{dt}=f_j(\varepsilon_{Lj-1},U_{Lj-1},\varepsilon_{Lj},U_{Lj}), && \frac{dU_{Lj}}{dt}=g_j(\varepsilon_{Lj-1},U_{Lj-1},\varepsilon_{Lj},U_{Lj}).\end{aligned}\right\} \qquad [21a–d]$$

Linear stability requires that the eigenvalues of the following matrix at steady state are all <0:

$$\left.\begin{aligned} &\frac{\partial(f_1,g_1)}{\partial(\varepsilon_{L1},U_{L1})}\\ &\frac{\partial(f_j,g_j)}{\partial(\varepsilon_{Lj-1},U_{Lj-1},\varepsilon_{Lj},U_{Lj})}\quad \text{for } j>1.\end{aligned}\right\} \qquad [22]$$

The eigenvalues are found by equating the following determinant to zero:

$$\Delta_M=\begin{bmatrix} \frac{\partial f_1}{\partial U_{L1}}-\lambda & \frac{\partial f_1}{\partial \varepsilon_{L1}} & 0 & 0 & 0 & 0 & \cdots\cdots\\ \frac{\partial g_1}{\partial U_{L1}} & \frac{\partial g_1}{\partial \varepsilon_{L1}}-\lambda & 0 & 0 & 0 & 0 & \cdots\cdots\\ \frac{\partial f_2}{\partial U_{L1}} & \frac{\partial f_2}{\partial \varepsilon_{L1}} & \frac{\partial f_2}{\partial U_{L2}}-\lambda & \frac{\partial f_2}{\partial \varepsilon_{L2}} & 0 & 0 & \cdots\cdots\\ \frac{\partial g_2}{\partial U_{L1}} & \frac{\partial g_2}{\partial \varepsilon_{L1}} & \frac{\partial g_2}{\partial U_{L2}} & \frac{\partial g_2}{\partial \varepsilon_{L2}}-\lambda & 0 & 0 & \cdots\cdots\\ 0 & 0 & \frac{\partial f_3}{\partial U_{L2}} & \frac{\partial f_3}{\partial \varepsilon_{L2}} & \frac{\partial f_3}{\partial U_{L3}}-\lambda & \frac{\partial f_3}{\partial \varepsilon_{L3}} & \cdots\cdots\\ 0 & 0 & \frac{\partial g_3}{\partial U_{L2}} & \frac{\partial g_3}{\partial \varepsilon_{L2}} & \frac{\partial g_3}{\partial U_{L3}} & \frac{\partial g_3}{\partial \varepsilon_{L3}}-\lambda & \cdots\cdots\\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots\cdots\\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots\cdots \end{bmatrix}. \qquad [23]$$

Note the special structure of this matrix—which has 2 × 2 matrices on the major diagonal and all terms to the right of it are zero. It can be shown that the value of the determinant Δ_M is the product of the determinants of the 2 × 2 matrices on the diagonal. All the 2 × 2 matrices on the diagonal are essentially identical with the exception of the first cell. This determinant will be designated as Δ_1 and the others Δ_j. The eigenvalues are the solution of

$$\Delta_M=\Delta_1\Delta_j^{n-1}=0, \qquad [24]$$

where

$$\Delta_1=\begin{vmatrix} -\frac{U_{LS}}{l\varepsilon_L}-\frac{4}{\rho_L D\varepsilon_L\sqrt{1-\varepsilon_L}}\frac{\partial\tau_{iL}}{\partial U_L}-\lambda & -\frac{U_{LS}^2}{l\varepsilon_L^3}+\frac{4}{\rho_L D\varepsilon_L\sqrt{1-\varepsilon_L}}\left(\frac{\partial\tau_i}{\partial\varepsilon_L}-\frac{\partial\tau_{iL}}{\partial\varepsilon_L}\right)\\ -\frac{\varepsilon_L}{l} & -\frac{U_{LS}}{\varepsilon_L l}-\lambda \end{vmatrix} \qquad [25]$$

and

$$\Delta_j=\begin{vmatrix} -\frac{U_{LS}}{l\varepsilon_L}-\frac{4}{\rho_L D\varepsilon_L\sqrt{1-\varepsilon_L}}\frac{\partial\tau_{iL}}{\partial U_L}-\lambda & \frac{\rho_G}{\rho_L}\frac{U_{GS}^2}{l(1-\varepsilon_L)^3}+\frac{4}{\rho_L D\varepsilon_L\sqrt{1-\varepsilon_L}}\left(\frac{\partial\tau_i}{\partial\varepsilon_L}-\frac{\partial\tau_{iL}}{\partial\varepsilon_L}\right)\\ -\frac{\varepsilon_L}{l} & -\frac{U_{LS}}{\varepsilon_L l}-\lambda \end{vmatrix}; \qquad [26]$$

equating Δ_1 and Δ_j to zero results in the following equations for the eigenvalues:

$$\Delta_1 = \lambda^2 + 2\left(\frac{U_{LS}}{l\varepsilon_L} + \frac{2\dfrac{\partial \tau_{iL}}{\partial U_L}}{\rho_L D \varepsilon_L \sqrt{1-\varepsilon_L}}\right)\lambda + \frac{4U_{LS}\dfrac{\partial \tau_{iL}}{\partial U_L}}{l\rho_L D {\varepsilon_L}^2 \sqrt{1-\varepsilon_L}} + \frac{4\left(\dfrac{\partial \tau_i}{\partial \varepsilon_L} - \dfrac{\partial \tau_{iL}}{\partial \varepsilon_L}\right)}{l\rho_L D\sqrt{1-\varepsilon_L}} = 0 \qquad [27]$$

and

$$\Delta_j = \lambda^2 + 2\left(\frac{U_{LS}}{l\varepsilon_L} + \frac{2\dfrac{\partial \tau_{iL}}{\partial U_L}}{\rho_L D \varepsilon_L \sqrt{1-\varepsilon_L}}\right)\lambda + \frac{4U_{LS}\dfrac{\partial \tau_{iL}}{\partial U_L}}{l\rho_L D {\varepsilon_L}^2 \sqrt{1-\varepsilon_L}} + \frac{4\left(\dfrac{\partial \tau_i}{\partial \varepsilon_L} - \dfrac{\partial \tau_{iL}}{\partial \varepsilon_L}\right)}{l\rho_L D\sqrt{1-\varepsilon_L}}$$

$$+\left(\frac{U_{LS}}{l\varepsilon_L}\right)^2 + \frac{\varepsilon_L \rho_G U_{GS}^2}{\rho_L l^2 (1-\varepsilon_L)^3} = 0. \qquad [28]$$

For stability, all values of λ should be <0. Since the coefficient of λ in the quadratic equations Δ_1 and Δ_j is always positive, the requirement for negative λ is satisfied when the zero-order term of the quadratic equations are >0, namely

$$c_1 = \frac{4}{l\rho_L D\sqrt{1-\varepsilon_L}}\left(\frac{U_{LS}}{\varepsilon_L^2}\frac{\partial \tau_{iL}}{\partial U_L} + \frac{\partial \tau_i}{\partial \varepsilon_L} - \frac{\partial \tau_{iL}}{\partial \varepsilon_L}\right) > 0 \qquad [29]$$

and

$$c_j = \frac{4}{l\rho_L D\sqrt{1-\varepsilon_L}}\left(\frac{U_{LS}}{\varepsilon_L^2}\frac{\partial \tau_{iL}}{\partial U_L} + \frac{\partial \tau_i}{\partial \varepsilon_L} - \frac{\partial \tau_{iL}}{\partial \varepsilon_L}\right) + \left(\frac{U_{LS}}{l\varepsilon_L}\right)^2 + \frac{\varepsilon_L \rho_G U_{GS}^2}{\rho_L l^2(1-\varepsilon_L)^3} > 0. \qquad [30]$$

The two last terms in [30] are positive, thus the need for $c_1 > 0$ satisfies also $c_j > 0$, hence the criterion for stability is

$$\frac{U_{LS}}{\varepsilon_L^2}\frac{\partial \tau_{iL}}{\partial U_L}\bigg|_{\varepsilon_L} + \frac{\partial \tau_i}{\partial \varepsilon_L}\bigg|_{U_{GS}} - \frac{\partial \tau_{iL}}{\partial \varepsilon_L}\bigg|_{U_L} > 0. \qquad [31]$$

Note that in [31] the differentiation of τ_{iL} is with respect to ε_L for constant U_L. Considering that $\tau_{iL} = f(U_{LS}(U_L, \varepsilon_L), \varepsilon_L)$ one can write the relation

$$\frac{\partial \tau_{iL}}{\partial \varepsilon_L}\bigg|_{U_L} = \frac{\partial \tau_{iL}}{\partial U_{LS}}\bigg|_{\varepsilon_L} \frac{\partial U_{LS}}{\partial \varepsilon_L}\bigg|_{U_L} + \frac{\partial \tau_{iL}}{\partial \varepsilon_L}\bigg|_{U_{LS}}. \qquad [32]$$

Substituting [32] in [31] and using the steady-state relation $U_{LS} = \varepsilon_L U_L$ yields the simple relation

$$\frac{\partial \tau_{iL}}{\partial \varepsilon_L}\bigg|_{U_{LS}} - \frac{\partial \tau_i}{\partial \varepsilon_L}\bigg|_{U_{GS}} < 0. \qquad [33]$$

This is exactly the same criterion as [15] which was obtained from the requirement of $C_K > 0$. This result is expected since the discrete form of the differential equation is consistent with the propagation of the kinematic waves in the downstream direction.

One of the simplified modes that has been suggested by Barnea & Taitel (1989) for analyzing the stability of the steady-state solutions, assumes a uniform film thickness along the pipe. This is a special case of the discrete form of the two-fluid model presented here, [17]–[20], where the number of cells, n, equals 1.

It has been shown that a linear stability analysis of the discrete form of the two-fluid model leads to a stability criterion, like that given in [15], independent of the number of cells. However, for a very fine discretization, where $n \to \infty$ and $l \to 0$, the zero-order term c_j of the quadratic equations for the eigenvalues [30] may attain a large value, leading to large values of the imaginary part of the eigenvalue. In this case, even if the steady state is stable (the real part of the eigenvalue is negative) the convergence to steady state is oscillatory.

The discrete form of the two-fluid model, will be used now to perform transient numerical simulations. These simulations can help us in analyzing the non-linear stability of the steady state and examining the transient response of the system to finite disturbances. It is shown here that although the original set of differential equations is ill-posed, the suggested discrete form enables one to perform transient numerical simulations around the steady-state solution and to get numerical results which seem to be physically logical.

TRANSIENT SIMULATIONS

Dynamic numerical runs were carried out using the discrete form of the two-fluid model, [17]–[20]. The transient simulations started at various deviations from the steady-state solutions, and the initial conditions were the same in all the cells. The results at different positions along the pipe are shown in figures 2–8. Two kinds of transient results are illustrated: (a) trajectories on a U_L vs ε_L plane; and (b) ε_L vs time.

Figures 2–5 show the dynamic response of the system to finite disturbances, for the case where a constant f_i is used and three steady-state solutions are obtained (see figure 1). It is clearly seen (figures 2–5) that the trajectories are attracted to the stable steady-state solution (A), even for relatively large deviations from the steady state. The convergence is smooth and without oscillations, in each of the cells along the pipe. A time delay in the response is noted as the observation point is moved further downstream in the pipe (figures 3 and 5).

Solution B, which is linearly unstable, is a repellent point and trajectories move away from this steady state even when the initial condition is exactly the steady-state value.

Point C, which was found to be linearly stable, is practically an unstable solution even for very small disturbances. Figures 2–5 show the dynamic response for a small deviation from steady state (C) for two values of the cell length (5 and 1 m). In the first cell, the system returns to steady state through small oscillations. However, as one moves downstream along the pipe these oscillations are amplified with substantial overshoots, where ε_L might exceed the value of 0.5 during the transient response. This will practically block the pipe cross section resulting in the collapse of annular flow. Note also that during the transient process the system passes through negative

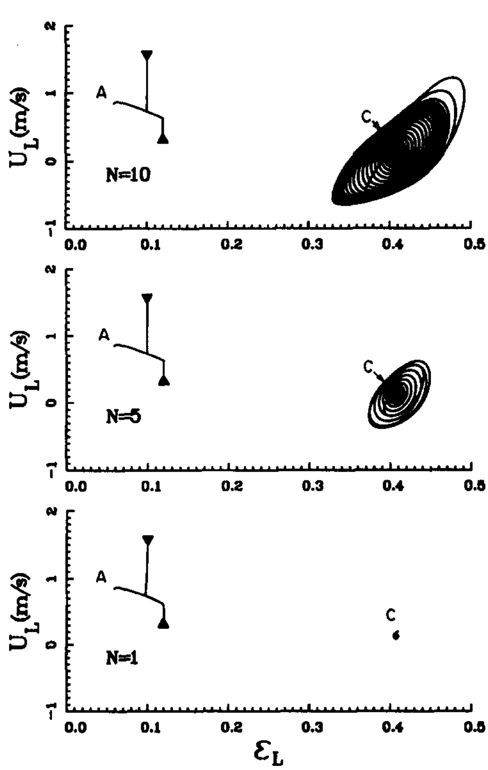

Figure 2. Dynamic simulations: U_L vs ε_L. Cell length $l = 5$ m (points A and C shown in figure 1).

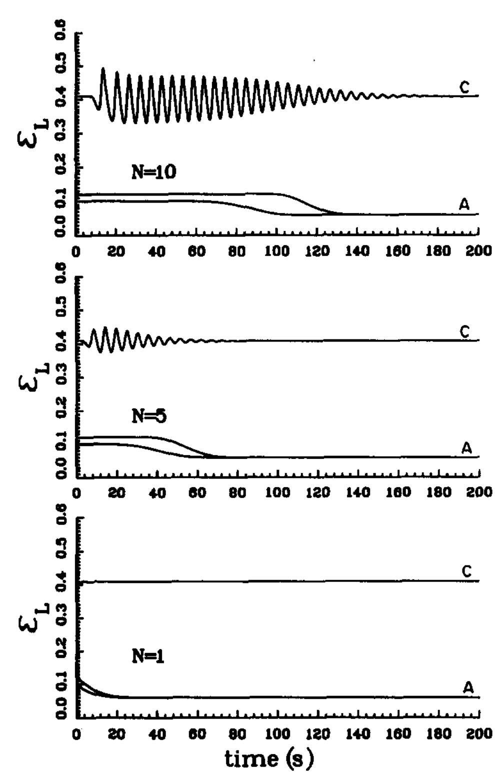

Figure 3. Dynamic simulations: ε_L vs time. Cell length $l = 5$ m (points A and C shown in figure 1).

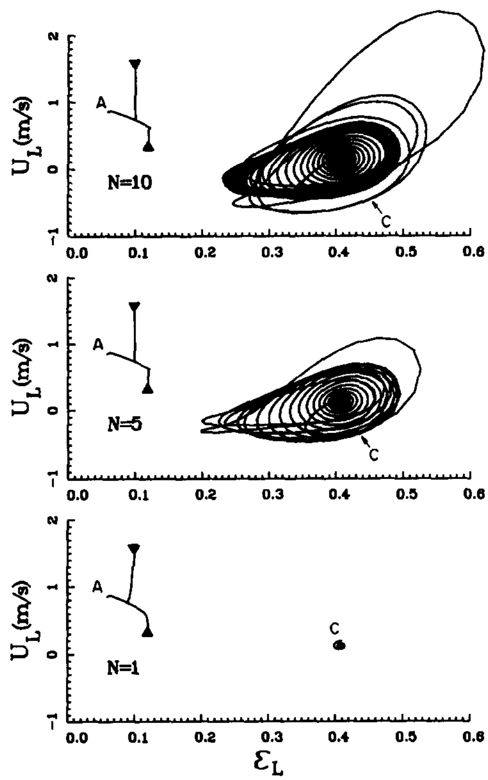

Figure 4. Dynamic simulations: U_L vs ε_L. Cell length $l = 1$ m (points A and C shown in figure 1).

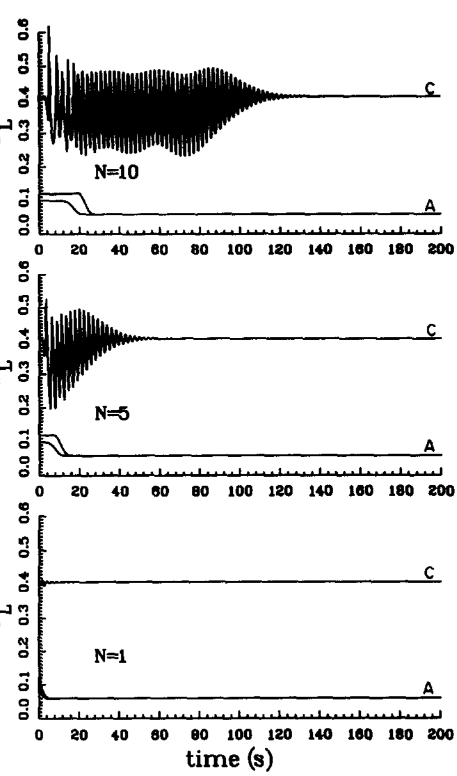

Figure 5. Dynamic simulations: ε_L vs time. Cell length $l = 1$ m (points A and C shown in figure 1).

liquid velocities which will destroy the structure of co-current annular flow. The finer the discretization is, the more pronounced and closer to the pipe entrance are the oscillations (figures 4 and 5).

Similar transient simulations have been carried out for the steady-state points D and E in figure 1. These steady states were obtained by using Wallis's (1969) correlation for f_i, and both, were found to be linearly stable. The behavior of point D, which is on the branch to the left of the minimum of the τ_{iL} vs δ curve, is similar to point A. The trajectories are attracted to point D in each of the cells along the pipe, for any grid size (figure 6). On the other hand, point E, although being linearly stable, has the same peculiar character as point C. The system returns to steady state (E) through substantial oscillations (figures 7 and 8) causing overshoots in the liquid holdup. Thus, practically only steady-state solutions that correspond to the thinnest film thickness are expected to exist (points A and D in figure 1).

SUMMARY AND CONCLUSIONS

The steady "two-fluid model" may yield multiple steady-state solutions for the average film thickness and phase velocities in steady annular flow. Our question of interest is which of these steady states are realized physically and will actually occur.

The respective transient formulation of the two-fluid model, [1]–[4], is ill-posed as an initial value problem and a linear stability analysis using this model indicates that the steady-state solutions are always unstable owing to the K–H nature of the instability. Namely, each solution of steady annular flow is dynamically unstable with respect to the wavy interface, due to the absence of gravitational stabilizing force.

However, our purpose is to study the stability of the structure of an annular flow with respect to its average film thickness and average film velocities as obtained by the steady-state solutions.

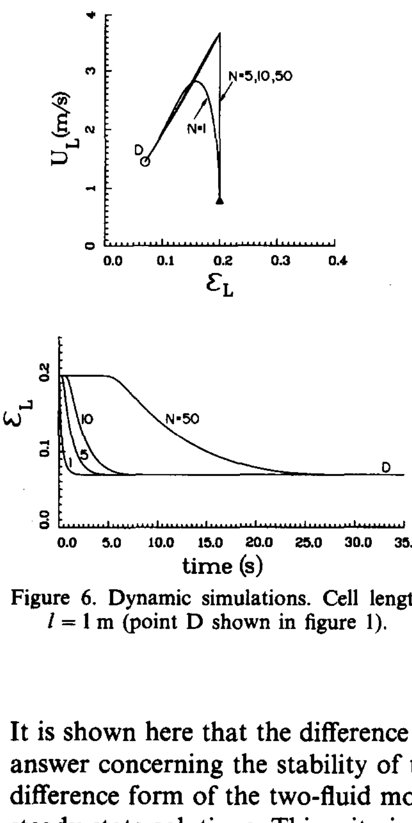

Figure 6. Dynamic simulations. Cell length $l = 1$ m (point D shown in figure 1).

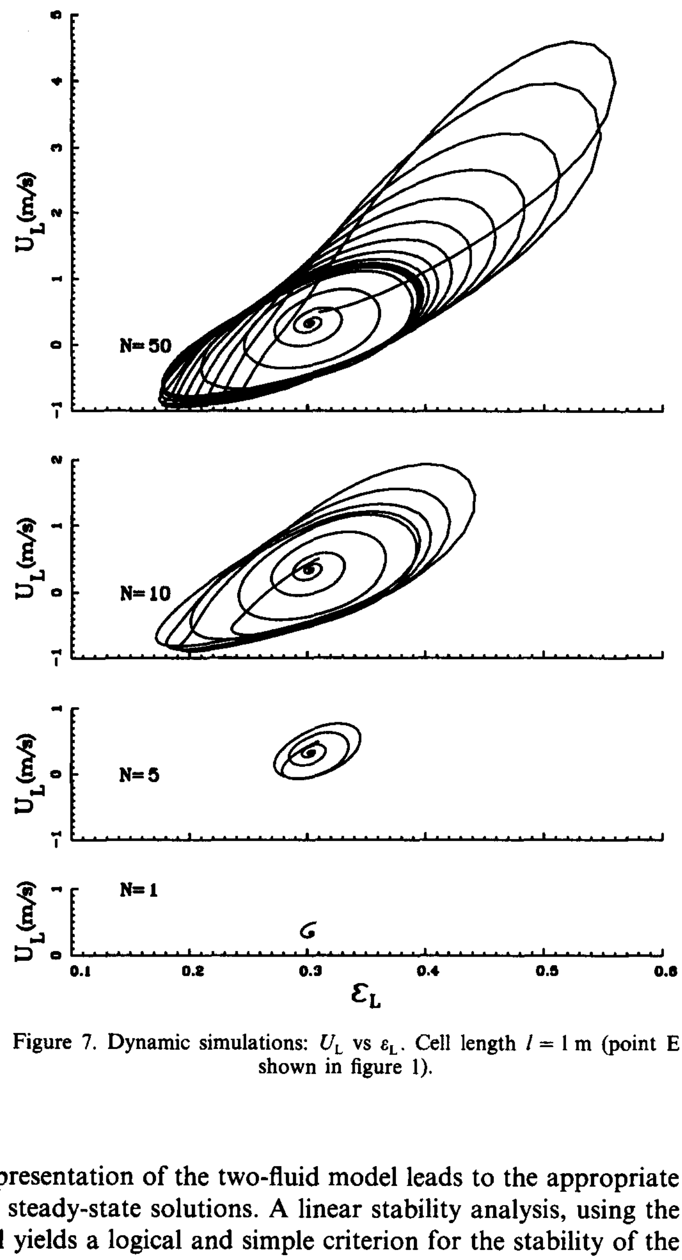

Figure 7. Dynamic simulations: U_L vs ε_L. Cell length $l = 1$ m (point E shown in figure 1).

It is shown here that the difference representation of the two-fluid model leads to the appropriate answer concerning the stability of the steady-state solutions. A linear stability analysis, using the difference form of the two-fluid model yields a logical and simple criterion for the stability of the steady-state solutions. This criterion is found to be consistent with the requirement that the K–H waves on the interface propagate in the downstream direction.

While the set of differential equations of the two-fluid model is ill-posed, the suggested discrete form can be used to carry out dynamic numerical simulations, which are used to examine the transient behavior of the system and the stability of the steady-state equations. It was found that only solutions that are associated with a thinner film are stable.

To summarize, one has to distinguish between the stability of the interface and the stability of the steady-state solutions, where the solution is obtained for an average film thickness using effective interfacial shear stresses. The stability of the interface is determined by a linear analysis using the differential equations [1]–[4]. While the answer to the question whether annular flow at a certain average film thickness is a stable structure is obtained by analysis of the discrete form of the "two-fluid model".

The relation between the stability of the steady-state solutions and the direction of propagation of the interfacial K–H waves may shed some light on the well-known dilemma, that equations that are ill-posed, can be used, in an appropriate scheme, to carry out transient simulations which converge to the stable steady states.

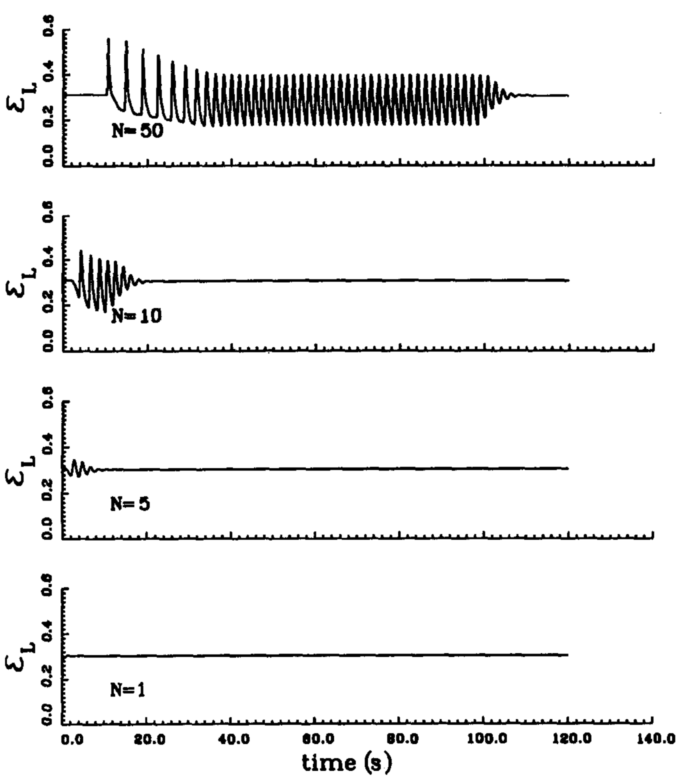

Figure 8. Dynamic simulations: ε_L vs time. Cell length $l = 5$ m (point E shown in figure 1).

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APPENDIX

The Numerical Cell Model

The pipe is subdivided into n cells of equal length, l. The flow is assumed to be incompressible with constant properties. Integral continuity and momentum balances are applied to the liquid and gas for each cell.

For cell j ($j > 1$) the following conservation equations are obtained for the liquid film:

$$\rho_L l \frac{dA_{Lj}}{dt} = \rho_L(U_{Lj-1}A_{Lj-1} - U_{Lj}A_{Lj}) \quad [A.1]$$

and

$$\rho_L l \frac{d(A_{Lj}U_{Lj})}{dt} + \rho_L(U_{Lj}^2 A_{Lj} - U_{Lj-1}^2 A_{Lj-1}) = -A_{Lj}\left(\frac{dP}{dz}\right)_j l - \tau_{Lj} S_{Lj} l + \tau_{ij} S_{ij} l - \rho_L A_{Lj} lg \sin\beta. \quad [A.2]$$

Since for the case of annular flow, the gas velocity is much faster than that of the liquid, the quasi-equilibrium condition is assumed for the gas:

$$U_{Gj-1}A_{Gj-1} = U_{Gj}A_{Gj} = U_{GS}A \quad [A.3]$$

and

$$\rho_G(U_{Gj}^2 A_{Gj} - U_{Gj-1}^2 A_{Gj-1}) = -A_{Gj}\left(\frac{dP}{dx}\right)_j l - \tau_{ij} S_{ij} l - \rho_G A_{Gj} lg \sin\beta. \quad [A.4]$$

Inserting [A.1] into [A.2] and rearranging yields:

$$\frac{dU_{Lj}}{dt} = \frac{1}{l}\left(\frac{U_{Lj-1}^2 A_{Lj-1}}{A_{Lj}} - \frac{U_{Lj-1}U_{Lj}A_{Lj-1}}{A_{Lj}}\right) - \frac{1}{\rho_L}\left(\frac{dP}{dz}\right)_j + \frac{\tau_{ij}S_{ij}}{\rho_L A_{Lj}} - \frac{\tau_{Lj}S_{Lj}}{\rho_L A_{Lj}} - g\sin\beta. \quad [A.5]$$

Inserting [A.3] into [A.4] yields

$$\frac{1}{l}\rho_G U_{GS}^2 A^2\left(\frac{1}{A_{Gj}^2} - \frac{1}{A_{Gj}A_{Gj-1}}\right) = -\left(\frac{dP}{dz}\right)_j - \frac{\tau_{ij}S_{ij}}{A_{Gj}} - \rho_G g \sin\beta. \quad [A.6]$$

Equating the pressure drop in [A.5] and [A.6] results in

$$\frac{dU_{Lj}}{dt} = \frac{A_{Lj-1}}{lA_{Lj}}(U_{Lj-1}^2 - U_{Lj-1}U_{Lj}) + \frac{\rho_G}{\rho_L}\frac{U_{GS}^2 A^2}{l}\left(\frac{1}{A_{Gj}^2} - \frac{1}{A_{Gj-1}A_{Gj}}\right)$$
$$- \frac{\tau_{Lj}S_{Lj}}{\rho_L A_{Lj}} + \frac{\tau_{ij}S_{ij}}{\rho_L A_{Lj}} + \frac{\tau_{ij}S_{ij}}{\rho_L A_{Gj}} - g\left(1 - \frac{\rho_G}{\rho_L}\right)\sin\beta. \quad [A.7]$$

Expressing the geometrical quantities in [A.1] and [A.7] in terms of $\varepsilon_L = A_L/A$ results in the following differential equations for cell j:

$$\frac{d\varepsilon_{Lj}}{dt} = \frac{1}{l}(U_{Lj-1}\varepsilon_{Lj-1} - U_{Lj}\varepsilon_{Lj}) \quad [A.8]$$

and

$$\frac{dU_{Lj}}{dt} = \frac{\varepsilon_{Lj-1}}{l\varepsilon_{Lj}}(U_{Lj-1}^2 - U_{Lj-1}U_{Lj}) + \frac{\rho_G}{\rho_L}\frac{U_{GS}^2}{l}\left[\frac{1}{(1-\varepsilon_{Lj})^2} - \frac{1}{(1-\varepsilon_{Lj-1})(1-\varepsilon_{Lj})}\right]$$
$$+ \frac{4}{\rho_L D\varepsilon_{Lj}\sqrt{1-\varepsilon_{Lj}}}(\tau_{ij} - \tau_{iLj}), \quad [A.9]$$

where

$$\tau_{iLj} = \left[g\left(1 - \frac{\rho_L}{\rho_G}\right)\sin\beta + \frac{4\tau_{Lj}}{\rho_L D\varepsilon_{Lj}}\right]\frac{\rho_L D\varepsilon_{Lj}\sqrt{1-\varepsilon_{Lj}}}{4}. \quad [A.10]$$

For the first cell, $j = 1$, the boundary conditions should be taken into account resulting in somewhat different equations. The contintuity and momentum equations for the liquid and gas are:

$$\rho_L \frac{d}{dt}(lA_{L1}) = \rho_L(U_{LS}A - U_{L1}A_{L1}), \qquad [A.11]$$

$$\rho_L \frac{d}{dt}(lA_{L1}U_{L1}) + \rho_L[U_{L1}^2 A_{L1} - (U_{LS}A)U_{Lin}] = -A_{L1}\left(\frac{dP}{dz}\right)_1 l - \tau_{L1}S_{L1}l + \tau_{i1}S_{i1}l - \rho_L A_{L1} lg \sin\beta; \qquad [A.12]$$

and

$$U_{G1}A_{G1} = U_{GS}A, \qquad [A.13]$$

$$\rho_G[U_{G1}^2 A_{G1} - (U_{GS}A)U_{Gin}] = -A_{G1}\left(\frac{dP}{dz}\right)_1 l - \tau_{i1}S_{i1}l - \rho_G A_{G1} lg \sin\beta. \qquad [A.14]$$

The inlet velocities, U_{Lin} and U_{Gin}, are taken as

$$U_{Lin} = \frac{U_{LS}A}{A_{L1}}$$

and

$$U_{Gin} = \frac{U_{GS}A}{A_{G1}}. \qquad [A.15]$$

Inserting [A.11] into [A.12] and [A.13] into [A.14] and equating the pressure drop in the gas and the liquid yields

$$\frac{dU_{L1}}{dt} + \frac{U_{LS}A}{A_{L1}l}\left(U_{L1} - \frac{U_{LS}A}{A_{L1}}\right) = -\frac{\tau_{L1}S_{L1}}{\rho_L A_{L1}} + \frac{\tau_{L1}S_{L1}}{\rho_L}\left(\frac{1}{A_{L1}} + \frac{1}{A_{G1}}\right) - \left(1 - \frac{\rho_G}{\rho_L}\right)g \sin\beta; \qquad [A.16]$$

or in terms of the liquid holdup:

$$\frac{dU_{L1}}{dt} = \frac{U_{LS}}{l\varepsilon_{L1}}\left(\frac{U_{LS}}{\varepsilon_{L1}} - U_{L1}\right) + \frac{4}{\rho_L D\varepsilon_{L1}\sqrt{1-\varepsilon_{L1}}}(\tau_{i1} - \tau_{iL1}), \qquad [A.17]$$

where

$$\tau_{iL1} = \left[g\left(1 - \frac{\rho_L}{\rho_G}\right)\sin\beta + \frac{4\tau_{L1}}{\rho_L D\varepsilon_{L1}}\right]\frac{\rho_L D\varepsilon_{L1}\sqrt{1-\varepsilon_{L1}}}{4}. \qquad [A.18]$$

Equation [A.17] together with the following continuity equation,

$$\frac{d\varepsilon_{L1}}{dt} = \frac{1}{l}(U_{LS} - \varepsilon_{L1}U_{L1}), \qquad [A.19]$$

are the representing equations for cell 1.